Proposition 7

The union of a countable collection of measurable set is measurable

Proof Let E be the union of countable collection of measurable sets $\{E_K\}_{K=1}^{\infty}$

Let $E_1' = E_1$

$E_2' = E_2 \sim E_1$

$E_3' = E_3 \sim (E_2 \cup E_1)$

$\vdots$

$E_K' = E_K \sim \left(\bigcup_{j=1}^{K-1} E_j\right)$ and so on.

Then $\bigcup_{K=1}^{\infty} E_K' = \bigcup_{K=1}^{\infty} E_K = E$ and E_K' are disjoint

$E_i' \cap E_j' = \phi \quad \forall i \neq j$.

E_K's are measurable for all $K = 1, 2, \ldots$

Thus E is an union of disjoint measurable sets.

To prove: E is measurable

ie To prove: for any set A, $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$

Let A be any set and let n be any natural number

Let $F_n = \bigcup_{K=1}^{n} E_K''$

since E_K' are measurable and since finite union of measurable sets is measurable.

F_n is measurable

$\therefore m^*(A) = m^*(A\cap F_n) + m^*(A\cap F_n^c)$

clearly $F_n \subset E$, $F_n^c \supset E^c$

$\therefore A\cap F_n^c \supset A\cap E^c$

$$m^*(A\cap E^c) \le m^*(A\cap F_n^c) \quad \text{——} ①$$

$$m^*(A) = m^*(A\cap F_n) + m^*(A\cap F_n^c)$$

$$\ge m^*(A\cap F_n) + m^*(A\cap E^c) \quad \text{——} ② \quad (\because \text{by } ①)$$

Now, $m^*(A\cap F_n) = m^*(A\cap (\bigcup_{K=1}^{n} E_K'))$

$$= \sum_{K=1}^{n} m^*(A\cap E_K')$$

$② \Rightarrow m^*(A) \ge \sum_{K=1}^{n} m^*(A\cap E_K') + m^*(A\cap E^c)$

Since n is arbitrary natural number.

$$m^*(A) \ge \sum_{K=1}^{\infty} m^*(A\cap E_K') + m^*(A\cap E^c) \quad \text{——} ③$$

$$A\cap E = A\cap (\bigcup_{K=1}^{\infty} E_K') \quad (\because E = \bigcup_{K=1}^{\infty} E_K = \bigcup_{K=1}^{\infty} E_K')$$

$$= \bigcup_{K=1}^{\infty} (A\cap E_K')$$

$$m^*(A\cap E) = m^*(\bigcup_{K=1}^{\infty} A\cap E_K')$$

$$\le \sum_{K=1}^{\infty} m^*(A\cap E_K')$$ [$\because$ by the union sub additive property of m^*]

$③ \Rightarrow m^*(A) \ge m^*(A\cap E) + m^*(A\cap E^c)$

Hence E is measurable

$[0,1]$ is not countable

soln

$m^*([0,1]) = 1 \neq 0$ ($\because$ Outer measure of an interval is its length)

since any countable set has outer measure zero

$\therefore [0,1]$ is not countable

---

let A be the set of irrational numbers in the interval $[0,1]$. Prove that $m^*(A)=1$

soln

let A be the set of irrational

let B be the set of rational numbers in $[0,1]$

Then B is countable

$$A \cup B = [0,1]$$

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

W.K.T $m^*(B) = 0$ [$\because$ countable set has outer measure zero]

$$m^*([0,1]) \leq m^*(A)$$

$$\text{ie, } 1 \leq m^*(A)$$

Also since $A \subseteq [0,1]$

$$m^*(A) \leq m^*[0,1] = 1$$

$$m^*(A) \leq 1$$

$$\text{ie, } m^*(A) = 1$$

Show that for any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$

soln

For any set E in R

$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) / E \subset I_k \right\}$

For any $\epsilon > 0$, there is an open cover $\{I_k\}$ of E such that $m^*(E) + \epsilon > \sum_{k=1}^{\infty} \ell(I_k)$

Take $\bigcup_{k=1}^{\infty} I_k = O$ a open set then $E \subset O$ then

$$m^*(O) = m^*\left(\bigcup_{k=1}^{\infty} I_k\right)$$

$$\leq \sum m^*(I_k)$$

$$= \sum_{k=1}^{\infty} \ell(I_k)$$

$$< m^*(E) + \epsilon$$

Now, O is open set, $E \subset O$ and $m^*(O) \leq m^*(E) + \epsilon$.

By the above part, for every natural number there exist an open set O_n such that $E \subset O_n$

and $m^*(O_n) < m^*(E) + \epsilon$

Then $G \in G_\delta$

$$m^*(O_n) \leq m^*(E) + \epsilon$$

$$\therefore m^*(G) \leq m^*(O_n) \leq m^*(E) + \epsilon \quad \forall \epsilon > 0$$

$$m^*(G) \leq m^*(E)$$

since $E \subset G$, $m^*(E) \subseteq m^*(G)$

$$\therefore m^*(G) = m^*(E)$$

G is a G_δ set

Let B be the set of rational numbers in the intervals $[0,1]$ and let $\{I_k\}_{k=1}^{\infty}$ be a finite subcollection of open intervals that covers B. Prove that $\sum_{k=1}^{\infty} m^*(I_k) \geq 1$

soln

$$B \subseteq \bigcup_{k=1}^{\infty} I_k$$

$$[0,1] \subset \bigcup \bar{I}_k$$

$$m^*([0,1]) \leq m^*(\bigcup \bar{I}_k)$$

$$\leq \sum m^*(\bar{I}_k)$$

$$= \sum m^*(\bar{I}_k)$$

$$\sum_{k=1}^{n} m^*(I_k) \geq 1$$

---

prove that if $m^*(A)=0$ then $m^*(A\cup B) = m^*(B)$

soln

$$m^*(A\cup B) \leq m^*(A) + m^*(B)$$

$$\therefore m^*(A\cup B) \leq m^*(B)$$

Also $B \subset A\cup B$

$$m^*(B) \leq m^*(A\cup B)$$

$$m^*(A\cup B) = m^*(B).$$

Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha \ \forall a \in A, b \in B$. Prove that $m^*(A\cup B) = m^*(A) + m^*(B)$.

soln

Case i) Suppose one of $m^*(A), m^*(B)$ is 0.

If $m^*(A)=0$

then $m^*(A\cup B) = m^*(B) = m^*(A) + m^*(B)$

If $m^*(B) = 0$

then $m^*(A\cup B) = m^*(A) = m^*(A) + m^*(B)$

($\because$ By above part)

case(ii) Suppose $m^*(A) \neq 0$ and $m^*(B) \neq \phi$

given that $\exists\ \alpha > 0$ such that $|a-b| \geq \alpha$, $\forall\ a\in A, b\in B$

we can choose countable collection of bounded open interval $\{I_K\}_{K=1}^{\infty}$ and $\{J_K\}_{K=1}^{\infty}$ such that

(i) $l(I_K) < \alpha/2$ and $l(J_K) < \alpha/2$ $\forall\ K = 1, 2, \ldots, l = 1, 2, \ldots$

(ii) $I_K \cap J_K = \phi$ $\forall\ K = 1, 2, \ldots$

(iii) $A \subseteq \bigcup_{K=1}^{\infty} I_K$ and $B \subseteq \bigcup_{K=1}^{\infty} J_K$.

Now, $A\cup B \subseteq \bigcup_{K=1}^{\infty} I_K \cup J_K$

$$m^*(A\cup B) = \inf\left\{\sum_{K=1}^{\infty} l(I_K\cup J_K) \,/\, A\cup B \subseteq \bigcup_{K=1}^{\infty}(I_K\cup J_K)\right\}$$

But $l(I_K\cup J_K) = l(I_K) + l(J_K)$ $[\because I_K\cap J_K = \phi]$

$$\sum_{K=1}^{\infty} l(I_K\cup J_K) = \sum_{K=1}^{\infty} l(I_K) + \sum_{K=1}^{\infty} l(J_K)$$

since $m^*(A) = \phi$, $\sum_{K=1}^{\infty} l(I_K) > 0$ for all possible open cover $\{I_K\}$ for A

similarly $m^*(B) \neq 0$, $\sum_{K=1}^{\infty} l(J_K) > 0$ for all possible open cover $\{I_K\}$ for B

$$\inf\left\{\sum_{K=1}^{\infty} l(I_K) + \sum_{K=1}^{\infty} l(J_K)\right\} = \inf\left\{\sum_{K=1}^{\infty} l(I_K)\right\} + \inf\left\{\sum_{K=1}^{\infty} l(J_K)\right\}$$

$$= m^*(A) + m^*(B)$$

$$m^*(A \cup B) = \inf \sum_{K=1}^{\infty} l(I_K \cup J_K)$$

$$= \inf\left(\sum_{K=1}^{\infty} l(I_K) + \sum_{K=1}^{\infty} l(J_K)\right)$$

$$= m^*(A) + m^*(B)$$

Remark

We have prove that union of countable collection of measurable set is measurable. Also from the definition of measurable set E^c is measurable. Therefore the set of all measurable set of all measurable set in a σ- Algebra

Note

By Demorgan's law intersection of the countable collection of measurable set is also measurable.

proposition 8

Every interval is measurable

proof since the collection of measure subset of R is σ - algebra

Let A be a σ-Algebra of subset of R containing the intervals of the form (a, ∞)

since A is a σ-algebra $(a,\infty) \in A$

$$(a,\infty)^c = (-\infty, a] \in A$$

$$[a,\infty) = \bigcap_{n=1}^{\infty} (a - \tfrac{1}{n}, \infty) \in A \text{ where } (a - 1/n, \infty) \in A \ \forall n = 1,2,\ldots$$

since A is a σ-algebra, countable intersection of member of A is also a member of A

A contains intervals of the form $[a,\infty)$

Again $[a,\infty)^c = (-\infty, a)$

A contains interval of the form $(-\infty, a)$

consider the following bounded intervals

$$[c,d] = (-\infty, d] \cap [c,\infty) \in A$$

$$[c,d) = (-\infty, d) \cap [c,\infty) \in A$$

$$(c,d) = (-\infty, d) \cap (c,\infty) \in A$$

$$(c,d] = (-\infty, d] \cap (c,\infty) \in A$$

Thus we have prove A contains all intervals

It is enough to prove that the interval (a,∞) is measurable

Let A be any set

Assume that $a \notin A$. if $a \in A$ then replace A by $A \sim \{a\}$

$(\because m^*(A \sim \{a\}) = m^*(A))$

$$A_1 = A \cap (-\infty, a), \ A_2 = A \cap (a, \infty)$$

claim: $m^*(A) \geq m^*(A_1) + m^*(A_2)$

ie, To prove: for any countable collection $\{I_K\}$ of open bounded interval that covers A.

It is enough to prove that $m^*(A_1) + m^*(A_2) \leq \sum_{K=1}^{\infty} \ell(I_K)$

consider $I_K' = I_K \cap (-\infty, a)$

$I_K'' = I_K \cap (a, \infty)$

$\therefore \{I_K'\}_{K=1}^{\infty}, \{I_K''\}_{K=1}^{\infty}$ are open bounded intervals

that covers A_1, A_2 respectively

ie, $A_1 \subseteq \bigcup_{K=1}^{\infty} I_K'$ and $A_2 \subseteq \bigcup_{K=1}^{\infty} I_K''$.

clearly $I_K = I_K' \cup I_K''$, $I_K' \cap I_K'' = \phi$

and hence $\ell(I_K) = \ell(I_K') + \ell(I_K'')$

$\therefore \sum_{K=1}^{\infty} \ell(I_K) = \sum_{K=1}^{\infty} \ell(I_K') + \sum_{K=1}^{\infty} \ell(I_K'')$

Now, $m^*(A_1) + m^*(A_2) \leq \sum_{K=1}^{\infty} \ell(I_K') + \sum_{K=1}^{\infty} \ell(I_K'')$

$= \sum_{K=1}^{\infty} \ell(I_K)$ [$\because$ Taking inf on the r.h.s]

$m^*(A_1) + m^*(A_2) \leq m^*(A)$

$\therefore (a, \infty)$ is measurable

Let B be the set of rational number in the interval $[0,1]$ and let $\{I_K\}_{K=1}^{n}$ be a finite collection of open intervals that covers B. Prove that $\sum_{K=1}^{n} m^*(I_K) \geq 1$

Since $\{I_K\}_{K=1}^{n}$ is a finite collection of open intervals that covers B.

$\bigcup_{K=1}^{\infty} I_K$ contains the set obtained from $[0,1]$ by omitting at most a finite number of irrationals

Let A be the set of finite number of irrationals omitted from $[0,1]$

Then $[0,1] - A \subseteq \bigcup_{K=1}^{n} I_K$

$$m^*([0,1]-A) \leq m^*\left(\bigcup_{K=1}^{n} I_K\right) \leq \sum_{K=1}^{n} m^*(I_K)$$

since $m^*([0,1]) = 1$ and $m^*(A) = 0$

$$m^*([0,1]-A) = 1$$

$$\sum_{K=1}^{n} m^*(I_K) \geq 1$$

Theorem:

A collection of M of measurable set is a σ-Algebra that contains the σ-Algebra $\mathcal{B}$ of Borel set Each interval each open set each closed set, each G_δ set and each F_σ set is measurable.

Proof: From the defn of measurable set, A set is measurable iff its complement is measurable.

$\therefore$ M is closed w.r.to the formation of complement. Also ϕ and $\mathbb{R}$ is measurable.

For any set A, (i) $m^*(A\cap\phi)+m^*(A\cap\phi^c) = m^*(A)+m^*(\phi)$

$= m^*(A)+0$

$= m^*(A)$

(ii) $m^*(A\cap R)+m^*(A\cap R^c) = m^*(A)+m^*(\phi)$

$= m^*(A)+0$

$= m^*(A)$

$\therefore \phi, R \in M$

Also we have proved that countable union of measurable set is measurable.

$\therefore M$ is a σ-Algebra

We have proved that every interval is measurable and hence being a countable union of disjoint open intervals.

A open set is measurable.

ie) M is σ-Algebra containing all open sets.

Since Borel σ-Algebra contained in every σ-Algebra that contains all open set

We have $\mathcal{B}$ is contained In M.

Since M is a σ-Algebra and since every interval is in M.

We have open set, closed sets and F_σ set and G_δ sets are measurable.

Defn: G_δ - A countable intersection of open sets

F_σ - countable union of closed sets.

A Borel σ-Algebra is contained in every σ-Algebra that contains all open sets in $\mathbb{R}$.

ie) The intersection of all σ-Algebra of a subset of $\mathbb{R}$ that contains a open set is a σ-Algebra is called Borel σ-Algebra.

The members of this collection are called Borel sets.

ie) Borel σ-Algebra contains all open set in $\mathbb{R}$.

Proposition: 10:

The translate of a measurable set is measurable.

proof: let E be a measurable set.

let A be any set and y be a real number.

To prove: $(E+y)$ is measurable

$$m^*(A) = m^*(A-y)$$

Since E is measurable,

$$m^*(A) = m^*(A-y)$$

$$m^*(A-y \cap E) + m^*(A-y \cap E^c)$$

since $A \cap E + y$ is translate of $A - y \cap E$ and $A \cap (E+y)^c$ is a translate of $A - y \cap E^c$

$$m^*(A - y \cap E) = m^*(A \cap (E+y)) \text{ and}$$

$$m^*(A - y \cap E^c) = m^*(A \cap (E+y)^c)$$

Hence $m^*(A) = m^*(A \cap (E+y)) + m^*(A \cap (E+y)^c)$

$\therefore (E+y)$ is measurable.

2.4 outer and inner approximation of lebesgue measurable sets:

If A is a measurable set of finite outer measure that is contained in B, then $m^*(B \sim A) = m^*(B) - m^*(A)$.

Proof: Now, $m^*(B) = m^*(B \cap A) + m^*(B \cap A^c)$

$$m^*(B) = m^*(A) + m^*(B \sim A)$$

$$m^*(B \sim A) = m^*(B) - m^*(A) \rightarrow (*)$$

The above property is called excision property.

Theorem: 11

Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E. [outer approximation by open sets and G_δ sets]

(i) For each $\varepsilon > 0$, there is an open set O containing E for which $m^*(O \sim E) < \varepsilon$

(ii) There is a G_δ sets containing E for which $m^*(G \sim E) = 0$

[Inner approximation by closed sets and F_σ sets]

(iii) For each $\varepsilon > 0$ there is an closed set F contained in E for which $m^*(E \sim F) < \varepsilon$

(iv) There is a F_σ set contained in E for which $m^*(E \sim F) = 0$.

Proof: Assume E is measurable

let $\varepsilon > 0$ be given

consider the case that $m^*(E) < \infty$

There is a countable collection of open bounded intervals collection $\{I_k\}_{k=1}^{\infty}$ which covers E for which $m^*(E) + \varepsilon > \sum_{k=1}^{\infty} l(I_k)$

Define $O = \bigcup_{k=1}^{\infty} I_k$

Then O is an open set set $E \subset O$ and

$$m^*(O) = m^*\left(\bigcup_{k=1}^{\infty} I_k\right)$$

$$\leq \sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} l(I_k)$$

$$< m^*(E) + \varepsilon$$

$$m^*(O) - m^*(E) < \varepsilon$$

Since E is measurable, $m^*(E) < \infty$,

$$m^*(O \sim E) < \varepsilon.$$

Suppose $m^*(E) = \infty$, Then E may be expressed as the disjoint union of countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable set, each set which has finite outer measurable.

ie) $E = \bigcup_{k=1}^{\infty} E_k$ and $m^*(E_k) < \infty \ \forall \ k = 1, 2, \ldots \infty$

Since E_K is measurable and $m^*(E_K) < \infty$

By case(i), for each K $\exists$ a open set O_K such that $O_K \supset E_K$ and $m^*(O_K \sim E_K) < \varepsilon/2^K$

Put $O = \bigcup_{K=1}^{\infty} O_K$, O is open se and $O \supset E$

$$O \sim E = \left(\bigcup_{K=1}^{\infty} O_K\right) - E$$

$$= \bigcup_{K=1}^{\infty} (O_K \sim E_K)$$

$$m^*(O \sim K) \le \sum_{K=1}^{\infty} m^*(O_K \sim E_K)$$

$$< \sum_{K=1}^{\infty} \varepsilon/2^K$$

$$= \varepsilon$$

$\therefore m^*(O \sim E) < \varepsilon$

Thus (i) holds in this case also.

(ii) For each K, choose an open set O_K such that $E \subset O_K$ and $m^*(O_K \sim E) < \frac{1}{K}$

Define $G = \bigcap_{K=1}^{\infty} O_K$

$\therefore G$ is a G_δ set such that $E \subset G$

$$m^*(G \sim E) \le m^*(O_K \sim E)$$

$$< 1/K$$

This is true $\forall K = 1, 2, \ldots$

$\therefore m^*(G \sim E) = 0$

Thus (ii) holds

We prove E is measurable

Since $m^*(G \sim E) = 0$ and a set of measure zero is measurable, $G \sim E$ is measurable.

since G is a G_δ set. G is measurable, we have $E = G \cap (G \sim E)^c$ measurable sets form a σ-Algebra.

(iii) Suppose E is measurable

Then E^c is measurable.

Hence for any $\varepsilon > 0$ there is an open set O such that $E^c \subset O$ for which $m^*(O \sim E^c) < \varepsilon$

consider $F = O^c$, a closed set and $F \subset E$

$$E \sim F = O \sim E^c$$

$$m^*(E \sim F) = m^*(O \sim E^c)$$

$$m^*(E \sim F) < \varepsilon$$

Thus F is a closed set such that $F \subset E$ and $m^*(E \sim F) < \varepsilon$.

(iv) Assume that E is measurable then E^c is measurable.

$\therefore$ (ii) holds goods for E^c

ie) $\exists$ a G_δ set G such that $E^c \subseteq G$ and $m^*(G \sim E^c) = 0$

let $F = G^c$ a F_σ set

clearly $G^c \subseteq E$

ie) $F \subseteq E$

$$E \sim F = E \sim G^c = G \sim E^c$$

$$m^*(E \sim F) = m^*(G \sim E^c) = 0$$

Thus $\exists$ a F_σ set such that $F \subseteq E$ and $m^*(E \sim F) = 0$.

Thus (iv) holds

We know that a set of measure zero is measurable and every F_σ set is measurable.

$\therefore E \sim F$ and F are measurable.

We can write $E = F \cup (E \sim F)$

Since the collection of measurable set is a σ-algebra we have E is measurable

Suppose (iii) holds good, then for each $\varepsilon > 0$ ∃ a closed set F such that $F \subseteq E$ and $m^*(E \sim F) < \varepsilon$

For each natural number k, ∃ a closed set $F_k \subset E$ such that $m^*(E \sim F_k) < 1/k$

$$\text{let } F = \bigcup_{k=1}^{\infty} F_k$$

Then F is a F_σ set and $F \subseteq E$

$$\therefore m^*(E \sim F) = m^*(E \sim \bigcup_{k=1}^{\infty} F_k)$$

Now, $E \sim (\cup F_k) \subseteq E \sim F_k$

$$m^*[E \sim (\cup F_k)] \leq m^*(E \sim F_k)$$

$$m^*[E \sim F] \leq m^*(E \sim F_k)$$

$$< 1/k \quad \forall\, k = 1, 2, \ldots$$

$$m^*(E \sim F) = 0$$

Thus (iv) holds good

Theorem 12:

Let E be a measurable set of finite outer measure then for each $\varepsilon > 0$ there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which $O = \bigcup_{k=1}^{n} I_k$. Then $m^*(E \sim O) + m^*(O \sim E) < \varepsilon$

Proof:

Since E is measurable and $m^*(E) < \infty$

There is a open set U such that $E \subset U$ and $m^*(U \sim E) < \varepsilon/2$

since E is measurable $m^*(U) - m^*(E) = m^*(U \sim E) < \varepsilon/2$

$\therefore m^*(U) < \infty$

since U is open, U is a countable collection of disjoint open intervals $\{I_k\}_{k=1}^{\infty}$

ie) $U = \bigcup_{k=1}^{\infty} I_k$

Hence for any n, $\sum_{k=1}^{n} l(I_k) = \sum_{k=1}^{n} m^*(I_k)$

$$= m^*\left(\bigcup_{k=1}^{n} I_k\right)$$

$$\leq m^*\left(\bigcup_{k=1}^{\infty} I_k\right)$$

$$= m^*(U) < \infty$$

$\therefore \sum_{k=1}^{\infty} l(I_k) < \infty \ \forall n.$

Hence $\sum_{k=1}^{\infty} l(I_k) < \infty$

This shows that that series $\sum_{k=1}^{\infty} l(I_k)$ converges, Hence there is a natural number n such that for given $\varepsilon > 0$,

$$\sum_{k=n}^{\infty} l(I_k) < \varepsilon/2$$

Define $O = \bigcup_{k=1}^{n} I_k$

$\therefore O \subset U$

Now, $O \sim E \subset U \sim E$

$\therefore m^*(O \sim E) \leq m^*(U \sim E) < \varepsilon/2$

$$E \sim O \subset U \sim O = \bigcup_{k=1}^{\infty} I_k \sim \bigcup_{k=1}^{n} I_k$$

$$= \bigcup_{k=n+1}^{\infty} I_k$$

$$\therefore m^*(E \sim O) \leq m^*\left(\bigcup_{k=n+1}^{\infty} I_k\right)$$

$$= \sum_{k=n+1}^{\infty} m^*(I_k)$$

$$= \sum_{k=n+1}^{\infty} l(I_k) < \varepsilon/2$$

$$m^*(E \sim O) + m^*(O \sim E) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Problem:

show that a set E is measurable iff for each $\epsilon>0$, there is a closed set F and open set O for which $F \subset E \subset O$ and $m^*(O \sim F) < \epsilon$.

Solution:

suppose E is measurable

Let $\epsilon > 0$ be given.

Then by equivalence of measurable $\exists$ an open set O and a closed set F such that $F \subset E \subset O$ and $m^*(O \sim E) < \frac{\epsilon}{2}$,

$m^*(E \sim F) < \epsilon/2$

$$m^*(O \sim F) = m^*(O \sim E) + m^*(E \sim F)$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

$m^*(O \sim F) < \epsilon$

conversely,

suppose there exist a open set O and a closed set F such that $F \subset E \subset O$ and $m^*(O \sim F) < \epsilon$ for given $\epsilon > 0$.

To prove: E is measurable.

Now, $O \sim E \subset O \sim F$.

$m^*(O \sim E) \leq m^*(O \sim F)$.

$< \epsilon_1$

$\therefore m^*(O \sim E) < \epsilon_1$

i.e) There exists a open set O such that $E \subset O$ and $m^*(O \sim E) < \epsilon$ for given $\epsilon > 0$.

This is the equivalent condition for measurability

$\therefore E$ is measurable.

Section: 2.5

countable Additivity continuity and the Borel–cantelli lemma.

Definition: The restriction of the set function outer measure to the class of measurable set is called Lebesgue measure. It is denoted by m.

If E is a measurable set, then its lebesgue measure $m(E)$ is defined by

$$m(E) = m^*(E).$$

Proposition: 13

Lebesgue measure m is countably additive.

Proof: Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable sets.

we have to prove $m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$.

This inequality is independent of n.

$$\sum_{k=1}^{\infty} m(E_k) \leq m\left(\bigcup_{k=1}^{\infty} E_k\right) \rightarrow (2).$$

From (1) and (2),

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

Theorem: 14

The set function lebesgue measure defined on the σ-algebra of lebesgue measurable sets assigns length to any interval is translation invariant and countable additive.

Proof: We know that every interval is measurable

$\therefore$ The lebesgue measure of an interval is the out measure.

But the outer measure of an interval its length.

$\therefore$ Lebegue measure of an interval is its length we proved that outer measure is translation invariant and the translate of a measurable set is also measurable.

Lebesgue measure is translation invariant we have already proved that lebesgue measure since countable union of measurable set is measurable.

$\bigcup_{k=1}^{\infty} E_k$ is measurable.

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$\leq \sum_{k=1}^{\infty} m^*(E_k)$$

But $m^*(E_k) = m(E_k)$

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) \rightarrow (1).$$

Now, $\bigcup_{k=1}^{n} E_k \subset \bigcup_{k=1}^{\infty} E_k$ for every natural number n.

By monotonicity property of m^*,

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq m^*\left(\bigcup_{k=1}^{\infty} E_k\right)$$

But $m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)$.

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

($\because E_k$'s measurable).

$$\therefore \sum_{k=1}^{n} m(E_k) \leq m^*\left(\bigcup_{k=1}^{\infty} E_k\right)$$

i,e) $\sum_{k=1}^{n} m(E_k) \leq m\left(\bigcup_{k=1}^{\infty} E_k\right)$ ($\because \bigcup_{k=1}^{\infty} E_k$ is measurable).

countably additive

Definition:

A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be ascending provided for each k, $E_k \subseteq E_{k+1}$ and said to descending provided for each k, $E_{k+1} \subseteq E_k$.

Theorem: (continuity of measure)

Lebesgue measure possess the following continuity proposition.

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable set then $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$ then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

proof

Given $A_1 \subset A_2 \subset \ldots$

(i) If there is a k_0 such that $m(A_{k_0}) = \infty$

then $m(A_k) = \infty \ \forall k \geq k_0$ [By monotonicity of measure).

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty \text{ and}$$

$$\lim_{k\to\infty} m(A_k) = \infty.$$

i.e) $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

Suppose $m(A_k) < \infty \ \forall k$.

Define : $A_0 = \phi$, $C_k = A_k \sim A_{k-1}$, $k = 1, 2, \ldots$

$\therefore \{C_k\}_{k=1}^{\infty}$ is a disjoint collection of measurable

sets and $\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} A_k$.

By the countable additive of lebesgue measurable

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right)$$

$$= \sum_{k=1}^{\infty} m(C_k)$$

$$= \sum_{k=1}^{\infty} m(A_k - A_{k-1})$$

$$= \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n \to \infty} [m(A_n) - m(A_0)] \quad (\because m(A_0) = 0).$$

$$= \lim_{n \to \infty} m(A_n) \ (\because m(A_0) = 0).$$

(ii) Again $B_1 \supset B_2 \supset \ldots$ and $m(B_1) < \infty$.

Define : $D_k = B_1 \sim B_k$.

$\therefore \{D_k\}$ is ascending family of measurable

with $m(D_k) < \infty \ \forall k$.

By the above part :

$$\therefore m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k \to \infty} m(D_k)$$

Now, $\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} (B_1 \sim B_k)$ $\left[\because \bigcup_{k=1}^{\infty} B_1 \sim B_k = \bigcup_{k=1}^{\infty} (B_1 \cap B_k^c) \right.$

$= B_1 \sim \bigcap_{k=1}^{\infty} B_k$ $\quad = \bigcup_{k=1}^{\infty} (B_1 \cap B_k^c)$

$= \left(B_1 \cap \bigcup_{k=1}^{\infty} B_k^c\right)$

$\therefore m\left(\bigcup_{k=1}^{\infty} D_k\right) = m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right)$ $\quad = B_1 \cap \left(\bigcup_{k=1}^{\infty} B_k^c\right)$

since each B_k is measurable $\quad = B_1 \cap \left(\bigcap_{k=1}^{\infty} B_k\right)^c$

and $B_k \subseteq B_1 \ \forall k$

by excision property, $\quad \left. = B_1 \sim \left(\bigcap_{k=1}^{\infty} B_k\right) \right]$

$m(B_1 \sim B_k) = m(B_1) - m(B_k)$ $(\because m(B_k) = m(B_1) < \infty)$

similarly, $\bigcap_{k=1}^{\infty} B_k$ is measurable and $\bigcap_{k=1}^{\infty} B_k \subseteq B_1$

By exision property,

$$m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$$

$$= \lim_{k\to\infty} [m(B_1 \sim B_k)]$$

$$= \lim_{k\to\infty} [m(B_1) - m(B_k)]$$

$$= m(B_1) - \lim_{k\to\infty} m(B_k)$$

$$\therefore m(B_1) < \infty - m\left(\bigcap_{k=1}^{\infty} B_k\right) = m(B_1) - \lim_{k\to\infty} m(B_k).$$

Since $m(B_1) < \infty$, $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$.

Definition:

For a measurable set E we say that a property holds almost everywhere on E, if it is holds for all $x \in E$ prove there is a subset E_0 of E for which $m(E_0) > 0$ and the property holds for all $x \in E \sim E_0$.

Lemma: (The Borel - Cantelli lemma)

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable set for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in R$ belong to almost finitely many of the E_k's.

Proof:

For any natural number n by the countable subadditive of m we have

$$m\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m(E_k) < \infty.$$

consider, $B_n = \bigcup_{k=1}^{n} E_k$.

$\therefore B_1 \supset B_2 \supset \ldots$ and $m(B_1) = m\left(\bigcup_{k=1}^{\infty} E_k\right)$

$$\leq \sum_{k=1}^{\infty} m(E_k)$$

$$< \infty.$$

By the continuity of Lebesgue measure

$$m\left(\bigcap_{k=1}^{\infty}\left(\bigcup_{k=1}^{n} E_k\right)\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k)$$

$$= 0 .$$

Now, $x \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$ if and only if x belongs to infinitely many E_k's . . . .

since $m\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right) = 0$ atmost all $x \in R$ fail to belongs to $\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$.

i.e) almost all $x \in R$ belongs to any finitely many E's.